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Stability Derivatives

What they are and how they are used

By Howard Loewen

Introduction

It was the mid-18th century when Leonhard Euler formulated his equations of motion for a rigid body. There are six equations – three force equations and three moment equations and, together, they provide enough information to fully describe the motion of a rigid body. If, for the time-being, we view an aircraft as a rigid body and disregard flexure, we can look to Eulers equations to help us understand basic aircraft dynamics. Pioneering studies in this field were conducted by G.H.Bryan who published a book "Stability in Aviation" in 1911. With very few changes, his treatment of the motion of an aircraft in 6-degrees of freedom is still in everyday use.

In the context of an aircraft, the forces and moments required by Eulers equations will be those generated by the reaction of air on the aircraft by virtue of its motion. Unfortunately, the functional form of aerodynamic forces and moments is not generally known so that mathematical devices need to be adopted to describe the relevant forces and moments. This is where stability derivatives come in.

This article begins by discussing stability derivatives as a way of representing aerodynamic forces and moments and explaining how they are derived. The article then takes an exemplar aircraft and puts forward detailed force and moment equations together with indications of which derivatives are important. Examples are presented showing how stability derivatives link to certain aspects of flight behaviour. The article concludes with advice on methods for estimating derivatives. It is emphasised that this article provides an introduction to the subject and is not a learned treatise.

Representation of aerodynamic forces and moments

The diagram shows an aircraft being acted upon by force and moment vectors (X,Y,Z) and (L,M,N).

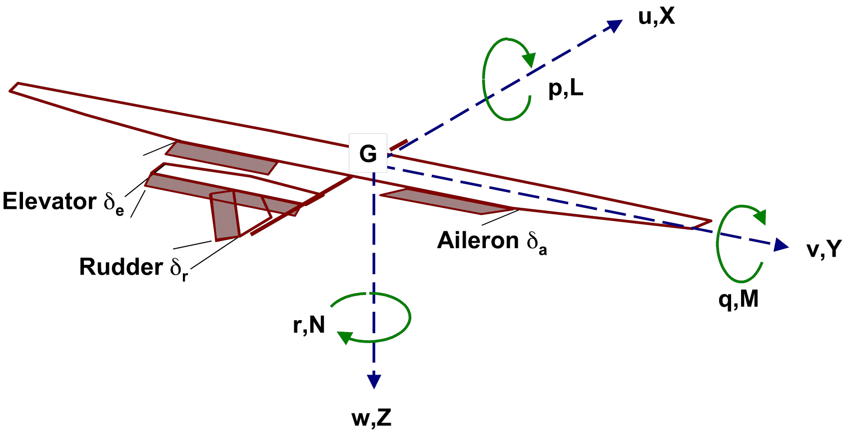

Figure 1 Notation

Instantaneously, the aircraft has linear and rotational velocity vectors (u,v,w) and (p,q,r) and the elevator, aileron and rudder deflections are (δ_e,δ_a,δ_r). Note that the axis system used here is body axes. There is another axis system much loved by aircraft aerodynamicists – wind axes – this system is briefly discussed later.

With the aircraft in this flight state, each of the force and moment terms will be expressible in the following general form:-

$$\textit{Force or moment} = f(u,v,w,p,q,r,\delta_e,\delta_a,\delta_r)$$

While this general form is valid, it is not actually very helpful. Over time, aerodynamicists have found it much more useful to describe the aerodynamic properties of an aircraft in terms of dimensionless parameters that are substantially independent of airframe size. For example:-

- it is known that forces and moments are proportional to dynamic pressure, Q, where $Q = \frac{1}{2}\rho V^2$ [ρ is air density and V is aircraft speed]
- to arrive at dimensionless parameters, forces and moments are normalised with respect to wing area (S_{WING}) and moments are also normalised with respect to a reference length which is conventionally wing chord (c) in the case of pitch and wing span (b) in the case of yaw and roll
- the lateral velocities w and v are converted into incidence, α, and sideslip β respectively, as follows:-

$$\alpha \approx w/V \text{ and } \beta \approx v/V$$

- body rates p, q and r are non-dimensionalised using division by (2V/reference length) where the reference length is equal to b or c as appropriate

If these substitutions are made, the equation for *f* above can be re-written, without loss of generality, into 3 force equations and 3 moment equations as follows:-

Force equations

$$X = QS_{WING}C_X\left(\alpha,\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)$$
$$Y = QS_{WING}C_Y\left(\alpha,\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)$$
$$Z = QS_{WING}C_Z\left(\alpha,\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)$$

Moment equations

$$L = QS_{WING}bC_l\left(\alpha,\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)$$
$$M = QS_{WING}cC_m\left(\alpha,\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)$$
$$N = QS_{WING}bC_n\left(\alpha,\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)$$

where (C_X, C_Y, C_Z) and (C_l, C_m, C_n) are now dimensionless functions known as aerodynamic coefficients.

The force and moment functions above are now in the nomenclature used by aerodynamicists but there is still the issue of how to translate those functions into meaningful descriptions of the forces and moments that can be used in analysis and modelling of aircraft dynamics. One option, which is appropriate to situations where high fidelity is required (such as flight simulators) would involve the use of multi-dimensional look-up tables to represent the aerodynamic coefficients. At the other extreme, these coefficients can be expanded in Taylor series where small perturbations are assumed about a reference flight state. In the context of UAVs, the look-up table approach is overkill while the Taylor series approach is more appropriate to UAV flight where the aircraft spends most of its time at or near a trimmed condition. In between, there is a third option in which the look-up table and Taylor series approaches are combined

To understand application of the Taylor series, we express the force and moment equations above collectively, as follows:-

$$y_i = d_i\left(z_j\right)$$

where $i = 1.......6$ represents the forces and moments (X,Y,Z,L,M,N) and $j = 1.....8$ represents the variables in the brackets on which the forces and moments depend. The Taylor series expansion with respect to a reference flight state, $\bar{z}_j$, then says the following:-

$$y_i = d_i\left(\bar{z}_j\right) + \left(\frac{\partial d_i}{\partial z_{j_1}}\right)\Delta z_{j_1} + \frac{1}{2}\left(\frac{\partial^2 d_i}{\partial z_{j_1}\partial z_{j_2}}\right)\Delta z_{j_1}\Delta z_{j_2} + ...$$

In this expansion, the partial derivatives, evaluated at the reference flight condition, are the ***aerodynamic*** or ***stability*** derivatives.

This intimidating-looking expansion has a parallel in the simple world of MS Excel where, if a polynomial trendline $a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ...$ is fitted to a function $y(z)$, the coefficients of *x* are analogous to the above terms in the Taylor expansion.

In practice, quadratic and cross-coupled terms associated with the double and higher derivatives in the Taylor series are very rarely needed so that stability derivatives are adequately represented by the first-order partial derivatives. These become the "placeholders" for the aerodynamic data of the aircraft under study. In this example, there are, theoretically, 6 x 8 = 48 "placeholders" for aerodynamic data but aerodynamicists have used their skill and judgement to establish that the ones that matter are much fewer in number.

For obvious reasons, stability derivatives are ***necessary*** to the study of aircraft dynamics. However, on their own, they are not ***sufficient*** because they need to be associated with a defined reference condition covering all the variables in the force and moment equations. In most instances, the logical reference values are obvious. For example, the aim is to fly aircraft without sideslip so the reference value of β would be zero. Similarly, under straight and level flight, body rates will be near-zero so that the reference values of *p,q,r* would also be zero. With equilibrium body rates set at zero, it is also logical to assign reference values of zero to rudder and aileron.

The interesting variables are incidence, α, and elevator deflection, δ_e. These are defined in different ways for different applications:-

- where a study is focussing on the basic dynamic properties of the aircraft and making the assumption of straight-and-level flight at constant speed (say), the analyst would calculate an (α,δ_e) pair to give trim for the speed of interest and use these to complete the set of reference values
- where a flight simulation is being performed, the (α,δ_e) pair is not known *a priori* and must be calculated by the simulation. A common way of addressing this problem is to get round it by making use of the third option mentioned above – mixing aerodynamic coefficients and stability derivatives in modelling the pitch plane normal force and moment. The dependence on incidence of normal force (Z) and pitching moment (M) is retained as a ***coefficient*** while the other terms, all of which have reference values of zero, are represented by ***stability derivatives***. Thus, the force equation (Z) becomes:-

$$Z = QS_{WING}\left[\underbrace{C_Z(\alpha)}_{coefficient} + \underbrace{C_Z{}'\left(\beta,\frac{pb}{2V_M},\frac{qc}{2V_M},\frac{rb}{2V_M},\delta_e,\delta_a,\delta_r\right)}_{stability\cdot derivatives}\right]$$

and similarly for M.

We now illustrate a practical set of stability derivatives using the aircraft shown below.

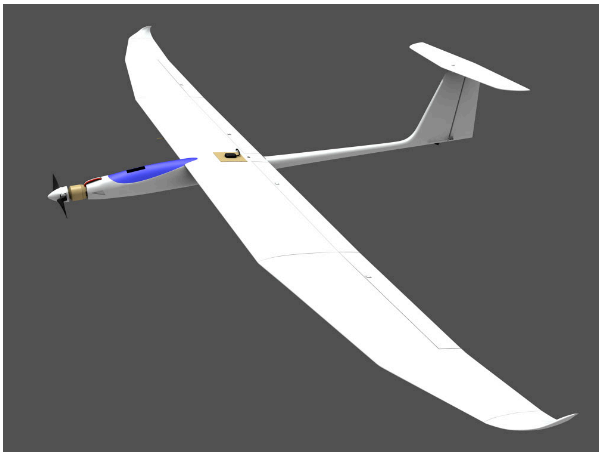

Figure 2 Cropcam UAV

We look first at longitudinal motion which depends on axial force (X), normal force (Z) and pitching moment (M). We then take the opportunity to discuss wind axes. We then look at lateral motion which depends on side force (Y), rolling moment (L) and yawing moment (N).

Longitudinal motion

In body axes, the equation for axial force, X, is very simple:-

$$X = QS_{WING}C_{d0}(\alpha)$$

where C_{d0} is known as the axial force coefficient. Note that this is actually a ***coefficient*** rather than a ***derivative*** though it is likely to have a dependence on incidence, α. The basic value of C_{d0} refers to the wing/body/tail combination with undeflected controls and, in line with the notation of Figure 1, will have a negative value. Additional terms may be needed to account for drag induced by deflection of the elevator, ailerons and rudder. However, the airframe shown has quite small controls so that the extra drag could probably be neglected.

A typical equation for the normal force, Z, is as follows:-

$$Z = QS_{WING}\left(C_Z(\alpha) + C_{Zq}\frac{qc}{2V_M} + C_{Z\delta e}\delta_e\right)$$

This equation further illustrates use of the "third way" to model normal force. Normal force generated by the wing/body/undeflected tail is represented by the coefficient $C_Z(\alpha)$ and most of this normal force will come from the wing. If the wing section is symmetric, normal force will be proportional to incidence but if the section is an aerofoil, as shown in Figure 3, there will be an additional (fixed) contribution to normal force that depends on the curvature of the camber line.

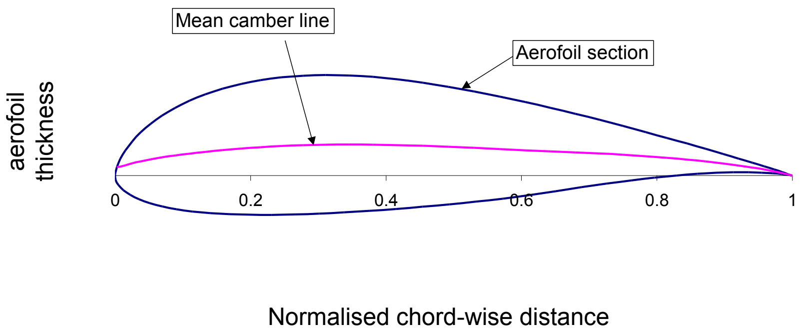

Figure 3 ***Mean camber line of an aerofoil***

Thus, in body axes, the equation for normal force can be re-written as follows:-

$$Z = QS_{WING}\left(C_{Z0} + C_{Z\alpha}\alpha + C_{Zq}\frac{qc}{2V_M} + C_{Z\delta e}\delta_e\right)$$

where C_{Z0} represents the camber effect and $C_{Z\alpha}$ represents the derivative with respect to incidence, α. For a practical airframe, all the terms in this equation are important with the exception of C_{Zq}. This derivative describes normal force due to pitch rate and is attributed mainly to incidence induced on the tail by aircraft rotation about its pitch axis. With a small tail area, this effect will be very small. It should be noted that, in line with the notation of Figure 1, normal force will be negative when an aircraft is in straight-and-level flight.

A typical equation for pitching moment, M, is as follows:-

$$M = QS_{WING}c\left(C_M(\alpha) + C_{Mq}\frac{qc}{2V_M} + C_{M\delta e}\delta_e\right)$$

The reference point for moments is the aircraft centre of gravity and the term $C_M(\alpha)$ describes the variation of pitching moment coefficient with incidence for the wing/body/undeflected tail while $C_{M\delta e}$ defines the pitching moment generated by elevator deflection. Since the centre of pressure of the wing will not, in general, coincide with the aircraft centre of gravity (it will usually be slightly behind), wing camber effect is again likely to introduce a finite pitching moment at zero incidence. Hence, the above equation can be re-written as follows:-

$$M = QS_{WING}c\left(C_{M0} + C_{M\alpha}\alpha + C_{Mq}\frac{qc}{2V_M} + C_{M\delta e}\delta_e\right)$$

Dynamic analysis using this equation shows that, with the elevator held fixed, the airframe will respond to a disturbance in the same way as a lightly damped spring/mass system. The undamped natural frequency is termed the "weathercock frequency" which can be estimated using the following formula (*J* is pitch inertia):-

$$\omega_{weathercock} = \sqrt{-\frac{QS_{WING}cC_{M\alpha}}{J}} \quad \text{rads/sec}$$

The damping ratio of the response will be determined by C_{Mq} and is likely to be less than 0.1. One aim of any autopilot will be to increase this damping by providing feedback from a pitch rate gyro.

The pitching moment equation is also central to trim analysis.

If one imagines an aircraft in steady straight-and-level flight, both pitch rate, *q*, and the net pitching moment (M) must be zero. It follows that:-

$$C_{M0} + C_{M\alpha}\alpha + C_{M\delta e}\delta_e = 0$$

Re-arranging this equation produces the following trim relationship:-

$$\delta_{TRIM} = -\frac{\left(C_{M0} + C_{M\alpha}\alpha_{TRIM}\right)}{C_{M\delta e}}$$

This equation is useful in helping to define the aerodynamic stability and elevator effectiveness required to achieve a certain trimmed incidence with a given elevator deflection limit.

One more parameter which is related to the pitch plane stability derivatives is the airframe "static margin". This is defined as the distance between where the normal force generated by the overall airframe acts and the airframe centre-of-gravity. If we take moments about the cg for the wing/body/undeflected tail for a fixed-incidence condition, we can say that the normal force (Z) × static margin (SM) must equal the pitching moment (M). We thus come up with the following relationship:-

$$\frac{SM}{c} = -\frac{\left(C_{M0} + C_{M\alpha}\alpha\right)}{\left(C_{Z0} + C_{Z\alpha}\alpha\right)}$$

The static margin is normalised with respect to wing chord and should be a negative fraction with magnitude << 1.

Wind Axes

The equations above have defined the pitch plane forces acting on the airframe (X and Z) in body axes but it is also common to find them expressed in wind axes which are defined in Figure 4 below.

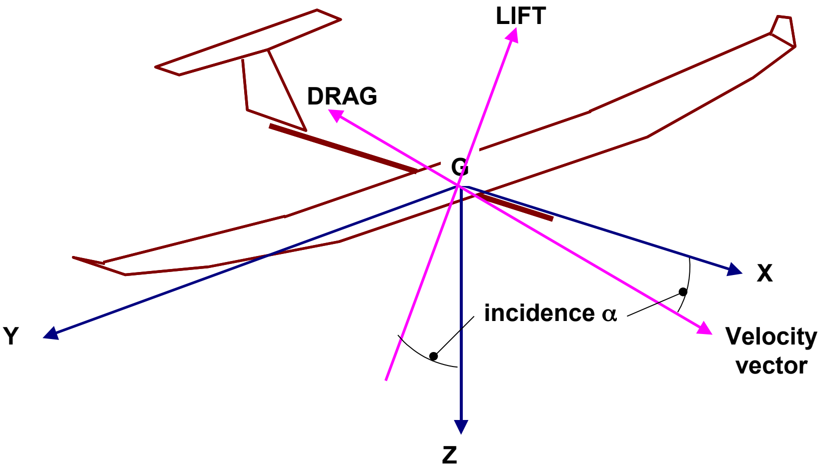

Figure 4 Wind axes defined

In wind axes, the x-direction is aligned with the velocity vector and drag is defined positive backwards. The y-direction is the same in wind and body axes but the z-direction is normal to the velocity vector and lift is defined positive upwards.

Stability derivatives can be defined in body or wind axes so it is clearly vital to know the reference axes. Body axes are best for dynamic analysis and modelling so if lift coefficient (C_L) and drag coefficient (C_D) are supplied, they can be transformed into body axes as follows:-

$$C_{D0} = C_L \sin\alpha - C_D \cos\alpha$$
$$C_Z = -C_L \cos\alpha - C_D \sin\alpha$$

Where the other derivatives are concerned, specialist student textbooks will define the transformations in full but the incidence effect is small and can probably be neglected. What cannot be neglected, however, is the effect on elevator-generated forces and moments. If these are defined in wind axes, they need to be negated before they are used in body axes.

Lateral motion

A typical equation for side force, Y, is as follows:-

$$Y = QS_{WING}\left(C_{Y\beta}\beta + C_{Yp}\frac{pb}{2V} + C_{Yr}\frac{rb}{2V} + C_{Y\delta r}\delta_r + C_{Y\delta a}\delta_a \right)$$

The purpose of the rudder is to keep sideslip to low levels and thus maintain the overall lift vector where it should be – normal to the wings. Since the fin is displaced from the yaw axis of the airframe, it is easy to imagine that yawing motion can induce local incidence on the fin and generate sideforce. Rolling motion can have the same effect since the centre of pressure of the fin will be displaced from the roll axis. However, these effects are generally negligible and, while aerodynamic prediction codes may generate numbers for all 5 derivatives, the reality is that $C_{Y\beta}$ and $C_{Y\delta r}$ are the only two derivatives out of the 8 that matter.

The equation for rolling moment, L, takes basically the same form as the side force equation, as follows:-

$$L = QS_{WING}b\left(C_{l\beta}\beta + C_{lp}\frac{pb}{2V} + C_{lr}\frac{rb}{2V} + C_{l\delta r}\delta_r + C_{l\delta a}\delta_a \right)$$

Aerodynamic prediction codes will generate numbers for all 5 derivatives but the 3 most important ones are $C_{l\beta}, C_{lp}, C_{l\delta a}$. $C_{l\beta}$ is sometimes referred to as the "dutch roll derivative" and is very important in determining the lateral static stability of the aircraft. Unfortunately, it is difficult to quantify since it depends on many factors including wing dihedral, wing sweep, wing/fuselage geometry and the fin geometry. C_{lp} represents the rolling moment due to roll rate and is important because it quantifies damping in roll. The moment comes about because, when the aircraft rolls, the incidence on one wing is increased which generates more lift and the incidence on the other wing is decreased generating less lift. The same effect occurs when the aircraft yaws but C_{lr} is much smaller in magnitude than C_{lp}. $C_{l\delta a}$ is of major importance since this represents the effect of deflecting the ailerons which is the primary roll control mechanism. A rolling moment can also be generated by deflecting the rudder ($C_{l\delta r}$) but, looking at Figure 2, it can safely be concluded that this effect will be much smaller than the aileron effect.

The equation for yawing moment, N, takes basically the same form as the rolling moment equation, as follows:-

$$N = QS_{WING}b\left(C_{n\beta}\beta + C_{np}\frac{pb}{2V} + C_{nr}\frac{rb}{2V} + C_{n\delta r}\delta_r + C_{n\delta a}\delta_a \right)$$

Not surprisingly, the fin plays a large, though not exclusive, part in determining the yawing moment derivatives. The derivative $C_{n\beta}$ is determined mainly by the fin characteristics and is important since it determines the weathercock frequency of the airframe in yaw which, in turn, quantifies the tendency of the aircraft to turn into wind following a sideslip disturbance. In yaw, the derivative C_{nr} provides weathercock damping analogous to C_{mq} in pitch. $C_{n\delta r}$ is of major importance since this represents the effect of deflecting the rudder. This is the primary yaw control mechanism. but there is potential for aileron deflection to generate a yawing moment; this comes about from differential drag between the left and right ailerons when they are deflected. Finally, the yawing moment due to roll rate, C_{np}, is also potentially significant. This is another moment that originates from differential drag generated by the wings as they roll.

Methods of derivative estimation

Knowing the background to stability derivatives and which ones are important is half the battle but the other half is finding methods of estimating the derivative values. There is a range of possibilities:-

- **specialist student textbooks –** these will contain simple derivations of analytical expressions. They will be useful in uncovering the physical origins of the derivatives (in more detail than above) but the estimates produced will be very approximate. Textbooks may also stop short of estimating aileron and rudder derivatives on the grounds that the prevailing aerodynamic conditions are surrounded by too much uncertainty
- **semi-empirical methods –** such as Digital DATCOM which was originally developed for the USAF but is now available as shareware. This prediction code relies on a combination of experimental data and aerodynamic theory; it thus offers the prospect of quite reliable estimates. It does, however, require a user with significant aerodynamic background.
- **Athena Vortex Lattice (AVL) program –** in this method, lifting surfaces are represented by many horseshoe vortices laid out in both spanwise and chordwise directions. This is a fairly computer-intensive technique but the only requirement on the user is to input the geometry of the aircraft. This program is also shareware. It cannot estimate drag but, in very limited testing, it has been found to estimate the other stability derivatives reasonably well.
- **Low speed wind tunnel tests –** this approach requires a modest investment in tunnel time but promises to produce the best estimates. If the tunnel has a fixed mounting for the aircraft model, only the static derivatives can be measured (those dependent on incidence, sideslip and control deflections). If

the aircraft model can be mounted on a flexible joint, the dynamic derivatives (those dependent on body rates) could also be measured. If such an arrangement is not available, “first cut” estimates of the dynamic derivatives might be generated using textbook methods

Schemes have also been synthesised for in-flight estimation of stability derivatives but these schemes are much less mature than the methods outlined above.

Conclusion

This article has explained the origins and some uses of aerodynamic stability derivatives in aircraft performance studies. Enough information has been provided to enable UAV designers/integrators to make a start on building mathematical models of their own UAVs. Once such a model is available, it places the UAV designer/integrator in the technically and commercially advantageous position of being able to “try-before-fly”.

On the technical side, desktop studies using the model would run alongside flight tests to provide some model validation. Once the model is a reasonable representation of the truth, desktop studies can be used to synthesise autopilot gains, etc, without an aircraft ever leaving the ground. Flight tests should then proceed with fewer problems. The controlled-airframe model would then be available for integration and testing in mission-type models.

The main commercial benefits that spring from being in a “try-before-fly” position with your UAV are reductions in technical risk and development/integration timescale and cost. Autopilot testing would take place initially on the ground where a wide range of test scenarios can be studied. There is no substitute for flight testing but greater use of desktop studies throughout a project should save time and money.

The “try-before-fly” formula has been used in the defence and civil aircraft industries for more than half a century and there is every reason for it to be successful in the context of UAVs.